

# GENERIC POINTS ON EXPONENTIAL CURVES

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ABSTRACT. We show, assuming Schanuel's conjecture, that every irreducible complex polynomial in two variables where both variables appear has infinitely many algebraically independent solutions of the form  $(z, e^z)$ .

## INTRODUCTION

A long-standing relevant conjecture in transcendence theory is Schanuel's conjecture, which states that given  $\mathbb{Q}$ -linearly independent complex numbers  $a_1, \dots, a_n$ , we have

$$\text{trdeg}_{\mathbb{Q}}(a_1, \dots, a_n, \exp(a_1), \dots, \exp(a_n)) \geq n.$$

This conjecture implies formally Lindemann-Weierstrass Theorem as well as the algebraic independence of  $e$  and  $\pi$ . In [11], Zilber noticed that Schanuel's conjecture could be interpreted in the setting of a predimension function à la Hrushovski. Put

$$\delta(A) := \text{trdeg}_{\mathbb{Q}}(A, \exp(A)) - \text{ldim}_{\mathbb{Q}} A,$$

for any finite subset  $A$  of  $\mathbb{C}$ . Then Schanuel's conjecture is equivalent to  $\emptyset$  being *strong* (or *self-sufficient*) in  $\mathbb{C}$ , that is,  $\delta(A) \geq 0$  for any finite dimensional  $\mathbb{Q}$ -subspace  $A$ . Zilber considered the language  $L$  of rings augmented by a unary function symbol  $\exp$  and  $L$ -structures (called *pseudo-exponential fields*) consisting of algebraically closed fields  $F$  of characteristic 0 equipped with a surjective group homomorphism  $\exp$  from its additive group onto its multiplicative group whose kernel is generated by a single transcendental element.

The above predimension function induces a finitary pregeometry in which given a finite subset  $A$  of  $F$ , its  $\delta$ -closure consists of all the elements  $b$  in  $F$  such that for some finite dimensional  $\mathbb{Q}$ -subspace  $B$  containing  $A \cup \{b\}$ ,

$$\delta(B/A) := \delta(B) - \delta(A) \leq 0.$$

Moreover, Zilber's fields  $F$  satisfy the following extra properties:

- (SC)  $\delta(A) \geq 0$  for any finite  $A \subseteq F$ .
- (CCP) The  $\delta$ -closure of any finite set is countable.

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- (EC) Given a variety  $V \subset F^{2n}$  defining a minimal extension of predimension 0, there is a generic point in  $V$  of the form  $(z, \exp(z))$ . Equivalently, there are infinitely many algebraically independent such points in  $V$ .

Though these conditions are not expressible in first-order logic, the class  $\mathcal{C}$  of pseudo-exponential fields is axiomatizable in  $\mathcal{L}_{\omega_1, \omega}(Q)$ , where  $Q$  denotes the quantifier “there exists uncountably many”. Zilber showed that  $\mathcal{C}$  is uncountably categorical, that is, there is a unique such field in each uncountable cardinal (up to isomorphism). Moreover, definable sets are either countable or co-countable. The question then is whether  $\mathbb{C}$  is the unique such field of cardinality  $2^\omega$ . By a clever use of Ax’s theorem [1] on Schanuel’s condition for the field of Laurent series in one variable, he concluded that  $\mathbb{C}$  already satisfies condition CCP. Note that since Schanuel’s conjecture is part of the axioms, the natural question is then the following:

**Question.** Assume  $\mathbb{C}$  satisfies SC. Does it follow that  $\mathbb{C}$  satisfies EC?

A warm-up case is when the variety is a curve in  $\mathbb{C}^2$  given by an irreducible complex polynomial  $p$  in two variables where both variables appear. Let  $f$  be the entire function given by  $f(z) = p(z, \exp(z))$ . In [10], Marker proved that such a function has infinitely many algebraically independent zeros if  $p$  is in  $\mathbb{Q}^{\text{alg}}[X, Y]$ .

In this article, we extend Marker’s result to all complex polynomials.

**Theorem.** *Suppose Schanuel’s conjecture is true. Then for an irreducible complex polynomial  $p$  in two variables where both variables appear, the entire function  $f(z) := p(z, \exp(z))$  has infinitely many algebraically independent zeros.*

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## 1. LINEAR RELATIONS IN FIELDS OF FINITE TRANSCENDENCE DEGREE

All throughout this section, let  $K$  be an algebraically closed subfield of  $\mathbb{C}$  of finite transcendence degree  $d$  containing  $\pi$ . Put  $\Gamma := \exp(K)$ , a subgroup of  $\mathbb{C}^\times$ . We consider solutions in  $\Gamma$  of

$$(\star) \quad \lambda_1 x_1 + \cdots + \lambda_k x_k = 1,$$

where  $\lambda_1, \dots, \lambda_k \in K$ . We say that a solution  $\vec{\gamma} = (\gamma_1, \dots, \gamma_k)$  in  $\Gamma$  of  $(\star)$  is *non-degenerate* if  $\sum_{i \in I} \lambda_i \gamma_i \neq 0$  for every nonempty proper subset  $I$  of  $\{1, \dots, k\}$ .

We begin with some notations that will be useful in the rest of the paper.

**Definition 1.1.** Let  $G$  be an abelian group, written multiplicatively and for  $n > 0$  put  $G^{[n]} = \{g^n : g \in G\}$ . We say that a subgroup  $H$  is *pure* in  $G$  if  $H \cap G^{[n]} = H^{[n]}$  for all  $n > 0$ . We say that  $H$  is *radical* in  $G$  if it is pure in  $G$  and it contains all the torsion elements of  $G$ .

Given  $A \subseteq G$ , we set  $\langle A \rangle_G$  to be the smallest radical subgroup of  $G$  containing  $A$ . That is,

$$\langle A \rangle_G = \{g \in G \mid g^n \in [A]_G \text{ for some } n \in \mathbb{N}\}$$

where  $[A]_G$  is the subgroup generated by  $A$ . When  $G$  is clear from the context, we will drop the subscripts and just write  $\langle A \rangle$  and  $[A]$ . For instance throughout this section the ambient group is  $\mathbb{C}^\times$  unless explicitly stated otherwise.

Since  $K$  is a field of characteristic 0, it is easy to see that  $\Gamma$  is divisible; in particular it is pure in  $\mathbb{C}^\times$ . Moreover  $\Gamma$  is a radical subgroup of  $\mathbb{C}^\times$  since  $\sqrt{-1}\pi$  is in  $K$ . Therefore, so is  $\Gamma \cap K^\times$ .

Given  $a_1, \dots, a_n$  in  $\mathbb{C}$ , by  $\vec{a}$  we denote the tuple  $(a_1, \dots, a_n)$  and  $\exp(\vec{a})$  denotes  $(\exp(a_1), \dots, \exp(a_n))$ .

Note the following straight-forward consequence of Schanuel's Conjecture concerning the rank of  $\Gamma \cap K^\times$ .

**Lemma 1.2.** *(Assuming Schanuel's conjecture)*

*The rank of  $\Gamma \cap K^\times$  is at most  $d$ .*

*Proof.* Let  $\beta_1, \dots, \beta_{d+1}$  in  $K$  such that  $\exp(\vec{\beta})$  is in  $\Gamma \cap K^\times$ . In particular,

$$\text{trdeg}_{\mathbb{Q}}(\beta_1, \dots, \beta_{d+1}, \exp(\beta_1), \dots, \exp(\beta_{d+1})) \leq d.$$

Thus Schanuel's conjecture implies that  $\beta_1, \dots, \beta_{d+1}$  are  $\mathbb{Q}$ -linearly dependent and hence  $\exp(\beta_1), \dots, \exp(\beta_{d+1})$  are multiplicative dependent.  $\square$

On the basis of this lemma, take  $\beta_1, \dots, \beta_t \in K$  where  $t \leq d$  such that  $\pi\sqrt{-1}, \beta_1, \dots, \beta_t$  are  $\mathbb{Q}$ -linearly independent and

$$\Gamma \cap K^\times = \langle \exp(\beta_1), \dots, \exp(\beta_t) \rangle.$$

Recall Lemma 8.2 from [3].

**Lemma 1.3.** *Let  $F$  be a field with a subfield  $E$  and subgroups  $G, H$  of  $F^\times$ . Suppose also that  $H$  is a radical subgroup of  $G$ . Then the following two conditions are equivalent:*

- (1) *for every  $\lambda_1, \dots, \lambda_k \in E$ , the equation  $(\star)$  has the same non-degenerate solutions in  $H$  as in  $G$ .*
- (2) *whenever  $g_1, \dots, g_n$  in  $G$  are multiplicatively independent over  $H$ , they are algebraically independent over the field  $E(H)$ .*

This allows us to prove the following.

**Proposition 1.4.** *(Assuming Schanuel's conjecture)*

*There exists a radical subgroup  $\Gamma^*$  of  $\Gamma$  of finite rank containing  $\Gamma \cap K^\times$  such that for every  $\lambda_1, \dots, \lambda_k$  in  $K$ , the equation  $(\star)$  has the same non-degenerate solutions in  $\Gamma^*$  as in  $\Gamma$ .*

*Proof.* Fix a transcendence basis  $\{\alpha_1, \dots, \alpha_s\}$  of  $K$  over  $\mathbb{Q}(\beta_1, \dots, \beta_t)$  and let  $e = \text{trdeg}_{\mathbb{Q}} K(\exp(\vec{\alpha}))$ . In particular,  $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_t$  are  $\mathbb{Q}$ -linearly independent and  $d \leq e \leq d + s$ .

By the previous lemma, we only need to construct a radical subgroup  $\Gamma^*$  of  $\Gamma$  of finite rank with the following property:

(\*) Let  $\gamma_1, \dots, \gamma_m \in \Gamma$  be algebraically dependent over  $K(\Gamma^*)$ . Then they are multiplicatively dependent over  $\Gamma^*$ .

We construct inductively an increasing chain of radical subgroups  $\{\Gamma_i\}_{i \in \mathbb{N}}$  of  $\Gamma$  of finite rank. Start with

$$\Gamma_0 := \langle (\Gamma \cap K^\times) \cup \exp(\vec{\alpha}) \rangle.$$

For  $i > 0$ , if there exist  $b_i^{(1)}, \dots, b_i^{(m_i)}$  in  $K$  such that  $\exp(\vec{b}_i)$  is multiplicatively independent over  $\Gamma_{i-1}$  but algebraically dependent over  $K(\Gamma_{i-1})$ , then let  $\Gamma_i :=$

$\langle \Gamma_{i-1} \cup \{\exp(\vec{b}_i)\} \rangle$ ; otherwise let  $\Gamma_i := \Gamma_{i-1}$ . (Note that the construction of the chain is not unique as it depends on the choices of the  $\vec{b}_i$ 's.)

We only need to show that the chain stabilises after  $i = d - t$ . Since at every step we add a tuple which is algebraically dependent over the previous one, we have that the transcendence degree of  $K(\Gamma_i)$  is at most  $e + (\text{rk}(\Gamma_i) - s - t) - i$ . In particular,

$$\text{trdeg}_{\mathbb{Q}} K(\Gamma_{d-t}) \leq \text{rk}(\Gamma_{d-t}).$$

Suppose there is some  $\vec{d}$  in  $K^n$  such that  $\exp(\vec{d})$  is algebraically dependent over  $K(\Gamma_{d-t})$ . Then the transcendence degree

$$\text{trdeg}_{\mathbb{Q}}(\vec{\alpha}, \vec{\beta}, \{\vec{b}_i\}_{1 \leq i \leq d-t}, \vec{d}, \exp(\vec{\alpha}), \exp(\vec{\beta}), \{\exp(\vec{b}_i)\}_{1 \leq i \leq d-t}, \exp(\vec{d}))$$

is at most  $\text{trdeg}_{\mathbb{Q}}(K(\Gamma_{d-t})) + n - 1$  and hence strictly less than  $\text{rk}(\Gamma_{d-t}) + n$ . Schanuel's conjecture yields a  $\mathbb{Q}$ -dependence among  $\vec{\alpha}, \vec{\beta}, \{\vec{b}_i\}_{1 \leq i \leq d-t}, \vec{d}$ . By the construction of the chain,  $\exp(\vec{d})$  is multiplicatively dependent over  $\Gamma_{d-t}$ , showing that  $\Gamma_{d-t} = \Gamma_i$  for  $i \geq d - t$  and that  $(*)$  is satisfied with  $\Gamma^* := \Gamma_{d-t}$ .  $\square$

**Remark.** It follows from the proof above that if the rank of  $\Gamma \cap K^\times$  is already  $d$ , then we can take  $\Gamma^*$  to be  $\Gamma \cap K^\times$ .

Let  $\Gamma^* = \langle \exp(a_1), \dots, \exp(a_s) \rangle$  with  $a_1, \dots, a_s \in K$  linearly independent over  $\sqrt{-1}\pi$ .

From now on,  $\mathbb{U}$  denotes the multiplicative group of all roots of unity. Recall the following results.

**Lemma 1.5.** (*Lemma 6.1 in [4]*)

Let  $E \subseteq F$  be fields such that  $E \cap \mathbb{U} = F \cap \mathbb{U}$  and  $G$  be a radical subgroup of  $E^\times$ . Then for  $\lambda_1, \dots, \lambda_n \in E^\times$ , the equation  $(*)$  has the same non-degenerate solutions in  $G$  as in  $\langle G \rangle_{F^\times}$ .

**Lemma 1.6.** (*Proposition 2.2 (ii) in [12]*)

Let  $L$  be a finitely generated extension of  $\mathbb{Q}(\mathbb{U})$ . Then the quotient group  $L^\times / \mathbb{U}$  is a free abelian group.

**Remark.** In [4], the statement of Lemma 1.5 is not quite correct. There  $G$  is taken to be a pure subgroup of  $E^\times$ , which is not enough to get the conclusion. However, it is easy to see that the proof there proves the lemma as stated here.

We can now reduce our situation from  $K$  to any subfield  $L$  that is finitely generated over  $\mathbb{Q}(\mathbb{U})$  containing the generators  $\exp(\vec{a})$ .

**Lemma 1.7.** Let  $L$  be a finitely generated extension of  $\mathbb{Q}(\mathbb{U})$  containing  $\exp(\vec{a})$ . Then there are  $c_1, \dots, c_{t'}$  in  $K$  linearly independent over  $\sqrt{-1}\pi$  such that for every  $\lambda_1, \dots, \lambda_k \in L$ , all the nondegenerate solutions of  $(*)$  in  $\Gamma^*$  are in  $\mathbb{U} \cdot [\exp(\vec{c})]$ .

*Proof.* In Lemma 1.5, take  $L, \mathbb{C}$  and  $\langle \exp(\vec{a}) \rangle_{L^\times}$  in the place of  $E, F$  and  $G$ . Note that then  $\langle G \rangle_{F^\times}$  is nothing other than  $\Gamma^*$ .

Being a divisible group,  $\mathbb{U}$  splits in  $\langle \exp(\vec{a}) \rangle_{L^\times}$  and consider its complement  $G'$ . Lemma 1.6 yields that  $G'$  is a subgroup of finite rank of a free commutative group, therefore  $G'$  is finitely generated, say  $G' = [\exp(\vec{c})]$ , with  $c_1, \dots, c_{t'} \in K$ . We may clearly assume that  $\vec{c}$  is linearly independent over  $\sqrt{-1}\pi$ .

Therefore the possible nondegenerate solutions in  $\Gamma^*$  of  $(\star)$  with  $\lambda_1, \dots, \lambda_k \in L$  are of the form

$$(\zeta_1 \exp(\vec{m}_1 \cdot \vec{c}), \dots, \zeta_k \exp(\vec{m}_k \cdot \vec{c}))$$

for some integer tuples  $\vec{m}_1, \dots, \vec{m}_k$  and roots of unity  $\zeta_1, \dots, \zeta_k$ .  $\square$

## 2. SPECIALIZATIONS AND REDUCTION TO A NUMBER FIELD

We first remark the following easy observation, whose proof follows the lines of the proof of Lemme 4 of [9] (Note that Laurent considered only finitely generated  $\mathbb{Q}$ -algebras, however his result is deeper).

**Lemma 2.1.** *Let  $R$  be a subring of  $\bar{\mathbb{Q}}[S]$ , where  $S$  is a finite subset of  $\mathbb{C}$ . Suppose that  $b_1, \dots, b_q$  are elements of  $R$  and let  $q'$  be the linear dimension over  $\bar{\mathbb{Q}}$  of  $\vec{b}$ . Then there are ring homomorphisms  $\phi_1, \dots, \phi_{q'}$  from  $R$  to  $\bar{\mathbb{Q}}$  fixing  $k := R \cap \bar{\mathbb{Q}}$  such that for every  $\alpha_1, \dots, \alpha_q$  in  $k$  with  $\alpha_1 b_1 + \dots + \alpha_q b_q \neq 0$  there is some  $i \in \{1, \dots, q'\}$  with  $\phi_i(\alpha_1 b_1 + \dots + \alpha_q b_q) \neq 0$ .*

*Proof.* After changing  $R$ , we may assume that  $q = q'$ . It suffices then to find  $\phi_i : R \rightarrow \bar{\mathbb{Q}}$  for each  $i \in \{1, \dots, q\}$  fixing  $k$  such that the determinant

$$D_q := \begin{vmatrix} \phi_1(b_1) & \cdots & \phi_1(b_q) \\ \vdots & & \vdots \\ \phi_q(b_1) & \cdots & \phi_q(b_q) \end{vmatrix}$$

is nonzero.

We proceed by induction on  $q$ . For  $q = 1$  by Nullstellensatz take a ring homomorphism  $R[b_1^{-1}] \rightarrow \bar{\mathbb{Q}}$  that fixes  $k$ . Clearly, its restriction to  $R$  sends  $b_1$  to some non-zero element.

Assume now that  $\phi_1, \dots, \phi_{q-1}$  have been already constructed such that  $D_{q-1} \neq 0$ . Then the determinant

$$D'_q := \begin{vmatrix} \phi_1(b_1) & \cdots & \phi_1(b_q) \\ \vdots & & \vdots \\ \phi_{q-1}(b_1) & \cdots & \phi_{q-1}(b_q) \\ b_1 & \cdots & b_q \end{vmatrix}$$

is  $\beta_1 b_1 + \dots + \beta_q b_q$ , where  $\beta_1, \dots, \beta_q$  are algebraic numbers. In particular, by induction,  $\beta_q = D_{q-1} \neq 0$ . Therefore, since we are assuming that the tuple  $\vec{b}$  is  $\bar{\mathbb{Q}}$ -linearly independent, we conclude that  $D'_q \neq 0$ . Consider

$$R' := R[(D'_q)^{-1}].$$

Nullstellensatz implies that there is a ring morphism  $\phi_q$  from  $R'$  to  $\bar{\mathbb{Q}}$  fixing  $k' := R' \cap \bar{\mathbb{Q}}$ . Its restriction to  $R$  has the property that  $\phi_q D'_q \neq 0$  which implies that  $D_q \neq 0$ .  $\square$

In order to reduce our setting to a number field in the last section, we need to carefully choose a specialization to  $\bar{\mathbb{Q}}$ . This is ensured by the density of closed points in specific subsets of the spectrum of any finitely generated  $\mathbb{Q}$ -algebra  $R$ . Given such  $R$  and a polynomial  $Q$  over  $R$  irreducible in  $\text{Frac}(R)[X]$ , denote by  $\Omega(Q)$  the collection of prime ideals  $\mathfrak{p}$  of  $R$  such  $Q \bmod \mathfrak{p}$  has the same degree as  $Q$  and it is irreducible as a polynomial over  $\text{Frac}(R/\mathfrak{p})$ . Recall that a *Hilbert set*  $\Omega$

is a subset of  $\text{Spec}(R)$  which contains a finite intersection of non-empty open sets and sets of the form  $\Omega(Q)$ .

**Fact 2.2.** Let  $R$  be a finitely generated  $\mathbb{Q}$ -algebra.

- (i) Given a finitely generated subgroup  $G$  of  $R^\times$ , there is a Hilbert set  $\Omega$  such that the residue map  $G \rightarrow (R/\mathfrak{p})^\times$  is injective for every  $\mathfrak{p}$  in  $\Omega$ .
- (ii) For any Hilbert set  $\Omega$  in  $R$ , the collection of maximal ideals contained in  $\Omega$  is dense in  $\text{Spec}(R)$ .

Combining the above with the proof of Lemma 2.1, one obtains the following result.

**Lemma 2.3.** (*Lemme 4 in [9]*)

Let  $R$  be a finitely generated  $\mathbb{Q}$ -algebra with largest subfield  $k$  and  $G$  a finitely generated subgroup of  $R^\times$ . Suppose also that  $b_1, \dots, b_q$  are elements of  $R$  that generate a  $\mathbb{Q}$ -linear space of dimension  $q'$ . Then there are ring homomorphisms  $\phi_1, \dots, \phi_{q'}$  from  $R$  into  $\bar{\mathbb{Q}}$  such that each  $\phi_i$  is injective on  $G$  and that for every  $\alpha_1, \dots, \alpha_q \in k$  with  $\alpha_1 b_1 + \dots + \alpha_q b_q \neq 0$ , there is  $i \in \{1, \dots, q'\}$  with

$$\phi_i(\alpha_1 b_1 + \dots + \alpha_q b_q) \neq 0.$$

In order to bound the degrees of the roots of unity appearing in Lemma 1.7 we will need the following result.

**Theorem 2.4.** (*Theorem 1 in [5]*)

Let  $F$  be a number field,  $a_0, a_1, \dots, a_k$  in  $F$  and  $\zeta$  a root of unity of order  $Q$  such that  $a_0 + \sum_{j=1}^k a_j \zeta^{n_j} = 0$  with  $\gcd(Q, n_1, \dots, n_k) = 1$ . Let  $\delta = [F \cap \mathbb{Q}(\zeta) : \mathbb{Q}]$  and suppose that for any nonempty proper subset  $I$  of  $\{0, 1, \dots, k\}$  the sum  $\sum_{j \in I} a_j \zeta^{n_j} \neq 0$ . Then for each prime  $p$  and  $n > 0$ , if  $p^{n+1} | Q$ , then  $p^n | 2\delta$  and

$$k \geq \dim_F(F + F\zeta^{n_1} + \dots + F\zeta^{n_k}) \geq 1 + \sum_{p|Q, p^2 \nmid Q} \left[ \frac{p-1}{\gcd(\delta, p-1)} - 1 \right].$$

In particular, the order  $Q$  of  $\zeta$  is bounded by a constant depending on  $k$  and  $\delta$  (and therefore  $[F : \mathbb{Q}]$ ).

The last result of this section concerns work from [8]. Work inside a number field  $F$ . For  $t, r$  in  $\mathbb{N}$  consider polynomials  $Q_1, \dots, Q_r$  over  $F$  in  $t$  many variables as well as a finite set  $Z := \{a_{ji} : j = 1, \dots, r; i = 1, \dots, t\}$  in  $F^\times$ . We are interested in describing the set of tuples  $\vec{m}$  in  $\mathbb{Z}^t$  such that

$$(**) \quad \sum_{j=1}^r Q_j(\vec{m}) \prod_{i=1}^t a_{ji}^{m_i} = 0.$$

For such an equation (\*\*), let  $H$  be the subgroup of those  $\vec{m}$  in  $\mathbb{Z}^t$  such that

$$\prod_{i=1}^t a_{ji}^{m_i} = \prod_{i=1}^t a_{j'i}^{m_i},$$

for every  $j, j' \in \{1, \dots, r\}$ .

Théorème 6 of [8] describes precisely the solutions of (\*\*), however for our purposes the following simplified version suffices.

**Theorem 2.5.** *Suppose that  $H$  is trivial. Then there are constants  $\delta, \eta$  depending only on  $Z$  and the field  $F$  such that if  $\vec{m}$  in  $\mathbb{Z}^t$  satisfies  $(**)$  and for every nonempty proper  $J \subseteq \{1, \dots, r\}$  the sum  $\sum_{j \in J} Q_j(\vec{m}) \prod_{i=1}^t a_{ji}^{m_i}$  is nonzero, then*

$$\|\vec{m}\| \leq \delta \log \|\vec{m}\| + \eta,$$

where  $\|\vec{m}\| := \max_i |m_i|$ .

**Remark.** The independence of the constants  $\delta, \eta$  from the coefficients of  $Q_i$  follows from the proof of [8]. Therefore, there is some  $N \in \mathbb{N}$  such that if  $\vec{m}$  satisfies a non-trivial equation  $(**)$ , then  $\|\vec{m}\| \leq N$ .

### 3. THE MAIN THEOREM

Here we prove the theorem stated in the introduction.

We keep the notations from the previous sections. In particular,  $K$  is an algebraically closed subfield of  $\mathbb{C}$  of finite transcendence degree containing  $\pi$  and the coefficients of  $p$ , which is an irreducible polynomial in two variables in which both variables appear.

Using Hadamard Factorization Theorem (see for instance [7]) and a result proved independently by Henson and Rubel [6] and by van den Dries [2], we have that  $f(z) = p(z, \exp(z))$  has infinitely many zeros in  $\mathbb{C}$  (for a proof of this, see [10]). Therefore in order to prove our theorem, it suffices to prove the following.

**Theorem 3.1.** *The entire function  $f(z) := p(z, \exp(z))$  has finitely many zeros in  $K$ .*

*Proof.* Write

$$p(X, Y) = \sum_{j=0}^m p_j(X) Y^j,$$

where  $p_j(X) \in K[X]$ . Also set  $I = \{j \in \{0, \dots, m\} \mid p_j \neq 0\}$ . Since  $p$  is irreducible, 0 lies in  $I$ . The set  $\{z \in \mathbb{C} \mid p_j(z) = 0 \text{ for some } j \in I\}$  is finite. Hence in order to show that there are finitely many solutions in  $K$  to  $p(z, \exp(z)) = 0$  we need only prove that

$$W := \{z \in K \mid p(z, \exp(z)) = 0 \wedge \bigwedge_{j \in I} p_j(z) \neq 0\}$$

is finite.

Let  $z$  be in  $W$ . By considering the appropriate subsum, we may assume that  $(\exp(z)^j)_{j \in I \setminus \{0\}}$  is a nondegenerate solution of

$$\sum_{j \in I \setminus \{0\}} -\frac{p_j(z)}{p_0(z)} x_j = 1.$$

Then by taking  $L$  to be an appropriate finitely generated extension of  $\mathbb{Q}(\mathbb{U})$  in Lemma 1.7, there are  $c_1, \dots, c_{t'}$  in  $K$  such that  $\exp(z)$  lies in the group

$$\mathbb{U} \cdot [\exp(c_1), \dots, \exp(c_{t'})].$$

Then

$$z \in \mathbb{Q}\pi\sqrt{-1} + \mathbb{Z}c_1 + \dots + \mathbb{Z}c_{t'}.$$

We now apply Theorem 2.4 to get a finer description of  $W$ .

**Claim.** *There is  $N \in \mathbb{N}$  such that if  $z \in W$  then there are  $k, l, m_1, \dots, m_{t'}$  in  $\mathbb{Z}$  and  $0 < n < N$  such that  $k < n$ ,  $\gcd(k, n) = 1$  and*

$$z = \frac{k2\pi\sqrt{-1}}{n} + l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j$$

*Proof.* Let  $z \in W \setminus \{0\}$  and choose  $k, l, m_1, \dots, m_{t'} \in \mathbb{Z}$  and  $n > 0$  such that  $k < n$ ,  $\gcd(k, n) = 1$ , and

$$z = \frac{k2\pi\sqrt{-1}}{n} + l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j.$$

We need to find a bound  $N$  on  $n$  independent of  $k, l, m_1, \dots, m_{t'}$ .

Set  $\vec{d} = \exp(\vec{c})$  and  $\zeta = \exp(2\pi\sqrt{-1}/n)$ . We then have

$$(*) \quad \sum_{j \in I} p_j \left( \frac{k2\pi\sqrt{-1}}{n} + l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j \right) \zeta^{kj} \cdot (\vec{d}^{\vec{m}})^j = 0.$$

Let  $R$  be the  $\mathbb{Q}$ -algebra generated by the coefficients of  $p$ ,  $\pi\sqrt{-1}$ ,  $\vec{c}$  and  $\vec{d}$  and their inverses. Using Lemma 2.1 with appropriate  $b_1, \dots, b_q$ , choose by Lemma 2.1 some specialization  $\phi$  such that

$$\phi(p_0 \left( \frac{k2\pi\sqrt{-1}}{n} + l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j \right)) \neq 0.$$

The homomorphism  $\phi$  transforms  $(*)$  into a non-trivial relation

$$\sum_{j \in I} \phi(p_j \left( \frac{k2\pi\sqrt{-1}}{n} + l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j \right)) \zeta^{kj} \cdot (\phi(\vec{d})^{\vec{m}})^j = 0.$$

Reorganizing we get a relation

$$\sum_{j \in I} a_j \zeta^{jk} = 0,$$

where the  $a_j$ 's are algebraic numbers depending on  $(n, k, l, \vec{m})$  and not all zero. Note however that the number field  $F$  containing the  $a_j$ 's is independent of  $(n, k, l, \vec{m})$ .

Let  $j_0 = \gcd(n, j)_{j \in I \setminus \{0\}}$  and  $\zeta_0 = \exp(\frac{2\pi\sqrt{-1}}{n/j_0})$ . So we have a relation

$$\sum_{j \in I} a_j \zeta_0^{\frac{jk}{j_0}} = 0.$$

For our purposes we may assume that no subsum is 0. Then Theorem 2.4 gives a bound on  $n$  depending only on the degree of  $F$  and  $|I|$ . Therefore there is  $N > 0$  depending only on  $p(X, Y)$  such that if  $(n, k, l, \vec{m})$  satisfies  $(*)$ , then  $n < N$ .  $\square$

Using this claim we may assume, after modifying  $f$  (finitely many times) that its zeroes in  $K$  are of the form

$$l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j$$



with  $l, m_1, \dots, m_{t'} \in \mathbb{Z}$ . Hence we have reduced the theorem to prove that there are only finitely many  $(l, \vec{m}) \in \mathbb{Z}^{1+t'}$  such that

$$(***) \sum_{j \in I} p_j(l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j)(\vec{d}^{\vec{m}})^j = 0 \text{ and } p_j(l2\pi\sqrt{-1} + \vec{m} \cdot \vec{c}) \neq 0 \text{ for } j \in I.$$

Let  $R$  be the  $\mathbb{Q}$ -algebra generated by the coefficients of  $p$ ,  $\pi\sqrt{-1}$ ,  $\vec{c}$ ,  $\vec{d}$  and their inverses. Let  $G$  be the multiplicative subgroup of  $R^\times$  generated by  $\vec{d}$ . Choose  $\phi_1, \dots, \phi_q$  ring homomorphisms from  $R$  to  $\mathbb{Q}$  injective on  $G$  as in Lemma 2.3 and let  $F$  be the compositum field of all their images.

Let  $(l, \vec{m})$  satisfy (\*\*\*) and choose  $\nu$  in  $\{1, \dots, q\}$  such that

$$\phi_\nu(p_0(l2\pi\sqrt{-1} + \sum_{j=1}^{t'} m_j c_j)) \neq 0.$$

The map  $\phi_\nu$  transforms (\*\*\*) into

$$\sum_{j \in I} p_{jl}(\vec{m}) \prod_{i=1}^{t'} \phi_\nu(d_i^j)^{m_i} = 0,$$

where  $p_{jl}(\vec{X})$  is a polynomial in  $(1+t')$ -variables such that

$$p_{jl}(\vec{m}) = \phi_\nu(p_j(l2\pi\sqrt{-1} + \vec{m} \cdot \vec{c})).$$

We may assume that no subsum is zero. Hence applying Theorem 2.5 and the remark after it, there is  $T$  in  $\mathbb{N}$  independent of  $l$  such that  $\|\vec{m}\| \leq T$ . The proof finishes by noting that for each  $\vec{m}$ , there are finitely many  $l$ 's satisfying (\*\*\*) □

**Question.** Do the techniques used in the proof of carry over to the case of a system of two polynomial equations?

$$p_1(x, y, \exp(x), \exp(y)) = 0$$

$$p_2(x, y, \exp(x), \exp(y)) = 0$$

It is not clear to the authors how to show that there are infinitely many solutions to the above system, since the proof of that depends heavily on Hadamard's factorization theorem [7] for one single complex variable.

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